Selected Topics in Probability (Spring 2021)
Homework Problems

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1 Random walks and electrical networks

1. (Reversible Markov chains and electrical networks)

   Let S be a finite or countably infinite set and let $P : S \times S \to [0,1]$ be the transition matrix of a Markov chain with state space S (i.e., $\sum_j P(i,j) = 1$ for each i). Assume that the chain is *irreducible* in the sense that for each $i, j \in S$ there exists a positive integer t such that $P^t(i,j) > 0$.

   The chain is called *reversible* if there exists a vector $\pi : S \to [0,\infty)$, $\pi \not\equiv 0$, such that
   $$\pi(i)P(i,j) = \pi(j)P(j,i) \quad \text{for all } i,j \in S. \tag{1}$$
   (the definition implies that π is a stationary vector for P. That is, $\pi P = \pi$ when π is considered as a row vector).

   The chain is *representable by an (electrical) network* if there exists $c : S \times S \to [0,\infty)$ (non-negative conductances), satisfying that $c(i,j) = c(j,i)$ and $c(i) := \sum_{k\in S} c(i,k) \in (0,\infty)$ for all $i,j \in S$, such that
   $$P(i,j) = \frac{c(i,j)}{c(i)} \quad \text{for all } i,j \in S. \tag{2}$$

   Prove that the chain is reversible if and only if it is representable by a network.

2. (Rough embedding and rough isometry)

   A *rough embedding* of a metric space (X_1, d_1) in another metric space (X_2, d_2) is a function $f : X_1 \to X_2$ satisfying that for real $\alpha \geqslant 1$, $\beta \geqslant 0$,
   $$\frac{1}{\alpha} d_1(x,y) - \beta \leqslant d_2(f(x), f(y)) \leqslant \alpha d_1(x,y) + \beta \quad \text{for all } x,y \in X_1. \tag{3}$$

   Two metric spaces (X_1, d_1) and (X_2, d_2) are said to be *rough isometric* if there exists a function $f : X_1 \to X_2$ and real $\alpha \geqslant 1$ and $\beta, \gamma \geqslant 0$ such that f satisfies (3) and, in addition,
   $$\text{for every } y \in X_2 \text{ there exists an } x \in X_1 \text{ such that } d_2(f(x), y) \leqslant \gamma. \tag{4}$$

   We may (and will) regard a graph $G = (V, E)$ as a metric space (V, d) by letting d be the graph distance between the vertices.

(a) Let G_1 be the infinite rooted binary tree (i.e., the tree with all vertices of degree 3 except the root which has degree 2). Let G_2 be the same graph with an additional infinite ray attached to the root. Prove that G_1 has a rough embedding in G_2 and G_2 has a rough embedding in G_1, but G_1 and G_2 are not rough isometric.

(b) Prove that the rough isometry relation is an equivalence relation on metric spaces.

Remark: As a first example, it is interesting to note that the lattice $\mathbb{Z}^d$ and the Euclidean space $\mathbb{R}^d$ are rough isometric.

Remark: Rough embedding and rough isometry are also called quasi-isometric embedding and quasi isometry.

3. (Rough embeddings and random walks; see definitions in previous problem)

   (a) Let G_1 be a bounded degree graph (i.e., there is a uniform bound to the degrees of all vertices) and let G_2 be a graph. Suppose that G_1 is transient and that G_1 has a rough embedding in G_2. Prove that G_2 is transient.

   Remark: In particular, all bounded degree graphs in a rough isometry equivalence class are simultaneously recurrent or simultaneously transient.

   (b) Find an example of a transient graph (necessarily of unbounded degree) which has a rough embedding in $\mathbb{Z}$.

The next three problems are taken from the chapter on random walks and electrical networks in the book of Asaf Nachmias.

4. (The Nash-Williams inequality is not necessary for recurrence)

   Let T be a rooted tree defined as follows: The root is the unique vertex at level 0 and it has two children. For each $n \geqslant 1$, the tree has 2^n vertices at level n, denoted $v_1^n, v_2^n, \ldots, v_{2^n}^n$, with the vertex v_k^n having one child if $1 \leqslant k \leqslant 2^{n-1}$ and having three children if $2^{n-1} < k \leqslant 2^n$.

   (a) Prove that T is recurrent.

   (b) Prove that $\sum_n |\Pi_n|^{-1} < \infty$ for any collection of disjoint edge cutsets Π_n separating the root from infinity (so that recurrence cannot be deduced from the Nash-Williams inequality).

5. (Effective resistance of dual planar graph)

   Let G be a finite *planar* graph with two distinct vertices a, z. Consider an embedding of G in $\mathbb{R}^2$ in which a is the leftmost vertex on the real axis, z is the right-most vertex on the real axis and both a and z lie on the outer face. Create a dual graph to G by first splitting the outer face by adding the rays from a to $-\infty$ and from z to $+\infty$ and then letting G^* be the dual of the resulting drawing (i.e., the dual vertices are the faces of the drawing). Write a^*, z^* for the vertices of G^* which correspond to the split outer face of G. Assume that all edge resistances are 1 and prove that

$$R_{\text{eff}}(a \leftrightarrow z; G) = \frac{1}{R_{\text{eff}}(a^* \leftrightarrow z^*; G^*)}. \tag{5}$$

6. Let $G = (V, E)$ be a graph with $V = \mathbb{Z}$ and with $E = \cup_{k=0}^{\infty} E_k$ where $E_0 := \{\{i, i+1\} \colon i \in \mathbb{Z}\}$ and

$$E_k := \left\{ \left\{ 2^k \left(n - \frac{1}{2}\right), 2^k \left(n + \frac{1}{2}\right) \right\} : n \in \mathbb{Z} \right\}. \tag{6}$$

Is G recurrent or transient? (provide a complete proof either way)

7. (Transient wedges in $\mathbb{Z}^3$)

Let $f : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$ be non-decreasing. Define the wedge W_f to be the induced subgraph of $\mathbb{Z}^3$ on the vertex set

$$\{(x, y, z) \colon x, y, z \geqslant 1,\ y \leqslant x \text{ and } z \leqslant f(x)\}. \tag{7}$$

Prove that W_f is recurrent if and only if $\sum_{k \geqslant 1} \frac{1}{k f(k)} = \infty$.

Remark: This result is due to Terry Lyons (1983).

2 Galton–Watson trees

The next two exercises analyze a critical Galton–Watson tree. The following are their common definitions and assumptions.

Let X be a random variable supported on $\{0, 1, 2, \ldots\}$. Consider a Galton–Watson tree with offspring distribution X and let Z_n be the number of descendants at level n (so that $Z_0 := 1$ and, conditioned on Z_{n-1}, Z_n is the sum of Z_{n-1} independent copies of X). We assume that the tree is critical, i.e., that $\mathbb{E}\, X = 1$ and $\mathbb{P}(X = 1) < 1$, and study the limiting behavior of Z_n following Kolmogorov 1938, Yaglom 1947 and Kesten–Ney–Spitzer 1966. We do so under the further assumption that $\operatorname{Var} X < \infty$.

Let $f(s) := \mathbb{E}\, s^X$ be the probability generating function of X, defined for $0 \leqslant s \leqslant 1$ (with $f(0) := \mathbb{P}(X = 0)$). Define, inductively, $f_0(s) := s$ and $f_n(s) := f(f_{n-1}(s))$ for $n \geqslant 1$, so that f_n is the probability generating function of Z_n.

1. (The asymptotics of f_n)

In this problem we prove that

$$\lim_{n \to \infty} \frac{1}{n} \left(\frac{1}{1 - f_n(s)} - \frac{1}{1 - s} \right) = \frac{\operatorname{Var} X}{2}, \quad \text{uniformly in } 0 \leqslant s < 1. \tag{8}$$

To this end, set $\alpha := \frac{\operatorname{Var} X}{2}$ and define a function $\varepsilon : [0, 1) \to \mathbb{R}$ by

$$f(t) := 1 + (t - 1) + \alpha(t - 1)^2 - \varepsilon(t)(t - 1)^2, \quad \text{for } 0 \leqslant t < 1. \tag{9}$$

Define also a function $\delta : [0, 1) \to \mathbb{R}$ by

$$\delta(t) := \frac{1}{1 - t} - \frac{1}{1 - f(t)} + \alpha. \tag{10}$$

(a) Prove that $0 \leqslant \varepsilon \leqslant \alpha$ for all t, that ε is a decreasing function with $\lim_{t \uparrow 1} \varepsilon(t) = 0$ and that

$$(1 - t)(\alpha - \varepsilon(t)) = \frac{f(t) - t}{1 - t} < 1. \tag{11}$$

Hint: You may prove first that $\varepsilon(t) = \mathbb{E} \left(\sum_{j=2}^{X-1} \sum_{v=1}^{j-1} (1 - t^v) \right)$.

(b) Prove that

$$-\alpha^2(1-t) \leqslant \delta(t) \leqslant \varepsilon(t) \quad \text{for } 0 \leqslant t < 1. \tag{12}$$

(c) Deduce from (12) that $\lim_{k\to\infty} \delta(f_k(s)) \to 0$ uniformly in $0 \leqslant s < 1$.

(d) Deduce (8) by considering $\frac{1}{n}\sum_{k=0}^{n-1} \delta(f_k(s))$.

2. (The limiting behavior of Z_n)

(a) (Survival probability) Deduce from (8) that

$$\lim_{n\to\infty} n \cdot \mathbb{P}(Z_n > 0) = \frac{2}{\operatorname{Var} X}. \tag{13}$$

Remark: In particular, the probability to survive for n levels is of order $\frac{1}{n}$ for critical Galton–Watson trees. Equation (13) holds also when $\operatorname{Var} X = \infty$ in the sense that the limit in the left-hand side is zero.

(b) (Distribution upon survival) Deduce from (8) that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{2Z_n}{n \operatorname{Var} X} > x \mid Z_n > 0\right) = e^{-x}, \quad \text{for all } x \geqslant 0. \tag{14}$$

Remark: In particular, the number of descendants at level n is of order n, when conditioned on surviving to level n (consistently with the fact that $\mathbb{E}\, Z_n = 1$). Determining the limiting behavior when $\operatorname{Var} X = \infty$ appears to be open.

Reminder: The uniqueness and continuity theorems for Laplace transforms: The Laplace transform of a probability measure μ supported in $[0,\infty)$ is the function $T_\mu : [0,\infty) \to (0,\infty)$ defined by $T_\mu(\lambda) := \int e^{-\lambda x} d\mu(x)$. (i) If $T_\mu = T_\nu$ then $\mu = \nu$. (ii) If $\lim_{n\to\infty} T_{\mu_n}(\lambda)$ exists for each $\lambda \geqslant 0$, and the limit is continuous at $\lambda = 0$, then there exists a probability measure μ such that μ_n converges weakly to μ (and, in particular, $T_{\mu_n} \to T_\mu$ pointwise).

Remark: Observe that the limiting behavior depends only on the variance of X - a form of universality.

3. (Generating uniform trees using Galton–Watson trees)

Let $n \geqslant 1$ be an integer. A *labelled* tree on n vertices is a graph on the vertex set $\{1,\ldots,n\}$ which is a tree. The number of such trees is n^{n-2} (this is Cayley's formula, originally proved by Borchardt) but this fact will not be needed here.

An *ordered* tree (or plane tree) is a rooted tree for which an ordering is specified for the children of each vertex.

(a) Let μ be the distribution on $\{0,1,2,\ldots\}$ with $\mu(k) = 2^{-(k+1)}$ (a type of geometric distribution). Let T be a Galton–Watson tree with offspring distribution μ, conditioned to have exactly n vertices; note that T is a random ordered tree. Prove that T is uniformly distributed over all ordered trees with n vertices.

(b) Let T be a Galton–Watson tree with offspring distribution Poisson(1), conditioned to have exactly n vertices. Do the following: (i) Assign the label 1 to the root of T, (ii) randomly assign the labels $2, \ldots, n$ to the other vertices of T via a uniform permutation and (iii) forget the ordering of T. Prove that the resulting unordered, labelled tree is uniformly distributed on all labelled trees with n vertices.

Remark: The above constructions allow to draw conclusions on uniformly-sampled labelled trees and uniformly-sampled ordered trees from the theory of critical Galton–Watson trees.

3 Stationarity and ergodicity

1. (Ergodicity of two-sided sequence)

   Let $(S, \mathcal{S})$ be a measurable space. Write $\varphi^{\mathbb{Z}}$ for the shift operation on $S^{\mathbb{Z}}$ and $\varphi^{\mathbb{N}}$ for the shift operation on $S^{\mathbb{N}}$ (here $\mathbb{N} = \{1, 2, \ldots\}$ and in both cases we mean that $\varphi((x_n)) = (x_{n+1})$). Write $\mathcal{I}^{\mathbb{Z}}$ and $\mathcal{I}^{\mathbb{N}}$ for the corresponding invariant sigma algebras (i.e., all events A satisfying that $\varphi^{-1}A = A$).

   Let $X = (X_n)_{n\in\mathbb{Z}}$ be a stationary sequence taking values in S. Write Y for its restriction to $\mathbb{N}$, i.e., $Y = (Y_n)_{n\in\mathbb{N}}$ defined by $Y_n = X_n$. Prove that X and Y are simultaneously ergodic (i.e., $\mathbb{P}(X \in A) \in \{0, 1\}$ for all $A \in \mathcal{I}^{\mathbb{Z}}$ if and only if $\mathbb{P}(Y \in B) \in \{0, 1\}$ for all $B \in \mathcal{I}^{\mathbb{N}}$).

   Hint: In one direction, show that for each $A \in \mathcal{I}^{\mathbb{Z}}$ there exists $B \in \mathcal{I}^{\mathbb{N}}$ such that $A = S^{\mathbb{Z}_{\leqslant 0}} \times B$ almost everywhere with respect to the distribution of X.

For the next two exercises, let $(\Omega, \mathcal{F})$ be a measurable space and $\varphi : \Omega \to \Omega$ a measurable map. Let $\mathcal{I}$ be the invariant sigma algebra (all $A \in \mathcal{F}$ with $\varphi^{-1}A = A$). Recall that a probability measure $\mathbb{P}$ on Ω is *preserved by* φ if $\mathbb{P}(\varphi^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$, and that such a $\mathbb{P}$ is called *ergodic* if

$$\mathbb{P}(A) \in \{0, 1\} \quad \text{for all } A \in \mathcal{I}. \tag{15}$$

2. (The probabilities of invariant events characterize stationary measures)

   Suppose $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures on Ω which are preserved by φ. Prove that if $\mathbb{P}_1(A) = \mathbb{P}_2(A)$ for all $A \in \mathcal{I}$ then $\mathbb{P}_1 = \mathbb{P}_2$.

   Hint: Use Birkhoff's ergodic theorem.

   Remark: A stronger fact is true (on Borel spaces): every stationary distribution can be decomposed uniquely as a mixture of ergodic distributions (the ergodic measures are the extreme points of the convex set of stationary measures).

3. (Tail triviality, mixing and ergodicity)

   A probability measure $\mathbb{P}$ on Ω (not necessarily preserved by φ) is said to have *trivial tail* if
   $$\mathbb{P}(A) \in \{0, 1\} \quad \text{for all } A \in \mathcal{T}, \tag{16}$$
   where the *tail sigma algebra* $\mathcal{T}$ consists of all events $A \in \mathcal{F}$ satisfying that for every $n \geqslant 1$ there is a $B_n \in \mathcal{F}$ with $A = \varphi^{-n}B_n$.

A probability measure $\mathbb{P}$ on Ω which is preserved by φ is called *mixing* if

$$\lim_{n\to\infty} \mathbb{P}(A \cap \varphi^{-n} B) = \mathbb{P}(A)\mathbb{P}(B) \quad \text{for all } A, B \in \mathcal{F}. \tag{17}$$

(a) Prove that a probability measure $\mathbb{P}$ on Ω (not necessarily preserved by φ) has trivial tail if and only if

$$\lim_{n\to\infty} \sup_{B\in\mathcal{F}} |\mathbb{P}(A \cap \varphi^{-n} B) - \mathbb{P}(A)\mathbb{P}(\varphi^{-n} B)| = 0. \tag{18}$$

Remark: If $\mathbb{P}$ is preserved by φ then $\mathbb{P}(\varphi^{-n} B) = \mathbb{P}(B)$, which simplifies (18) and makes it stronger than the mixing property (17).

(b) Let $\mathbb{P}$ be a probability measure on Ω which is preserved by φ. Prove that $\mathbb{P}$ is ergodic if and only if

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{P}(A \cap \varphi^{-k} B) \to \mathbb{P}(A)\mathbb{P}(B) \quad \text{for all } A, B \in \mathcal{F}. \tag{19}$$

Remark: We thus see that (when $\mathbb{P}$ is preserved by φ), tail triviality is stronger than mixing and mixing is stronger than ergodicity. The next part shows that the latter inclusion is strict and it can also be shown that the former inclusion is strict.

(c) Recall the rotation of the circle transformation: $\Omega = [0, 1]$, $\varphi(\omega) = \omega + \theta$ (mod 1) for an irrational θ and $\mathbb{P}$ being Lebesgue measure on Ω. Prove that $\mathbb{P}$ is not mixing.

4 Subadditive ergodic theorem and first-passage percolation

1. (Last-passage percolation on a tree / maximum of branching random walk)

   Let T be a binary tree with root vertex o (root of degree 2, all other vertices of degree 3). To each edge e of T associate an independent real-valued random variable η_e, all having a common distribution ν. For each vertex v of T, let S_v be the sum of the η_e along the edges of the path from o to v. Let D_n be the set of vertices at distance n from v. Set

   $$M_n := \max\{S_v \colon v \in D_n\} \tag{20}$$

   to be the last-passage time from o to level n (or, with a different interpretation, the maximum at level n of a branching random walk).

   Assume that

   $$\mathbb{E}(e^{\lambda X}) < \infty \text{ for all real } \lambda \text{ when } X \text{ is a random variable distributed as } \nu. \tag{21}$$

   (a) Prove that $\mathbb{E}(M_n) < \infty$.

   (b) Prove that there exists a deterministic constant x for which

   $$\frac{M_n}{n} \to x \text{ almost surely and in } L^1. \tag{22}$$

(c) Prove that

$$x \leqslant \sqrt{2\log 2} \tag{23}$$

when ν is the standard normal distribution.

Remark: In fact, $x = \sqrt{2\log 2}$ (for the standard normal distribution) and this can be proved by a "modified" second moment argument. By modified we mean that the basic approach of calculating the second moment of the number of $v \in D_n$ with large S_v does not suffice in this case and one needs to modify it by considering those $v \in D_n$ which satisfy additional restrictions. For details see, e.g., the lecture notes of Zeitouni on branching random walks and Gaussian fields.

2. (Dekking–Host argument for tightness)

We continue with the last-passage percolation on a tree setup introduced in the previous problem. Suppose, in addition to (21), that the distribution of ν has mean zero and is supported on $(-\infty, 1]$.

(a) Prove that

$$\mathbb{E}(M_{n+1} - M_n) \leqslant 1.$$

(b) Write $e_1 = (o, v_1)$, $e_2 = (o, v_2)$ for the two edges incident to the root vertex. When w is a descendant of v, write S_w^v for the sum of the η_e along the edges of the path from v to w. Let

$$M_n^1 := \max\{S_w^{v_1} : w \in D_{n+1},\ w \text{ is a descendant of } v_1\},$$
$$M_n^2 := \max\{S_w^{v_2} : w \in D_{n+1},\ w \text{ is a descendant of } v_2\},$$

be the last-passage times from v_1 and v_2, respectively, to their descendants at level $n+1$.

Observe that $M_{n+1} = \max\{M_n^1 + \eta_{e_1}, M_n^2 + \eta_{e_2}\}$. Use this fact and the identity $\max\{a,b\} = \frac{1}{2}(a + b + |a-b|)$ to deduce that

$$\mathbb{E}(M_{n+1} - M_n) \geqslant \frac{1}{2}\,\mathbb{E}\,|M_n - M_n'| \tag{24}$$

where M_n' is an independent copy of M_n.

(c) Deduce from the previous two parts that the sequence $(M_n - \mathbb{E}(M_n))$ is tight. That is,

$$\lim_{M\uparrow\infty} \sup_n \mathbb{P}(|M_n - \mathbb{E}(M_n)| > M) = 0.$$

Remark: The last part shows that the fluctuations of the last-passage time on a binary tree are of order at most 1 (and it is simple to see that they cannot be of even lower order).

3. (Poissonian directed last-passage percolation)

Let P be a Poisson process of unit intensity in the non-negative orthant $[0,\infty)^d$, $d \geqslant 2$. For $x \in [0,\infty)^d$, denote by $\mathcal{P}_x$ the collection of paths in $[0,\infty)^d$ from the origin to x which are non-decreasing in every coordinate (that is, the set of continuous maps $p : [0,1] \to [0,\infty)^d$ with $p(0) = (0,\dots,0)$ and $p(1) = x$ such that $p(t)-p(s) \in [0,\infty)^d$ for every $0 \leqslant s \leqslant t \leqslant 1$). Define T_x to be the maximal number of points of P on a path of $\mathcal{P}_x$ (the maximum is realized on a piecewise linear path).

(a) Prove that for every $x \in [0,\infty)^d$ the limit

$$\mu(x) := \lim_{n\to\infty} \frac{1}{n} T_{n\cdot x}$$

exists almost surely and in L^1 (with the limit over positive integer n).

(b) Let $\lambda := \mu(1,1,\ldots,1)$. Find explicit $\lambda_0, \lambda_1 \in (0,\infty)$, depending on the dimension d, such that $\lambda_0 < \lambda < \lambda_1$.

(c) Use the symmetries of the Poisson process to prove that

$$\mu(x) = \lambda \cdot (x_1 x_2 \cdots x_d)^{1/d} \quad \text{for } x = (x_1,\ldots,x_d) \in [0,\infty)^d.$$

Remark: In dimension $d = 2$, Vershik–Kerov (1977) and Logan–Shepp (1977) proved that $\lambda = 2$. Later, two-dimensional Poissonian directed last-passage percolation became one of the central examples of an integrable model of last-passage percolation following the breakthrough work of Baik–Deift–Johansson (1999). A beautiful introduction to the topic is in the book "The surprising mathematics of longest increasing subsequences" by Dan Romik.

5 Ising and Spin $O(n)$ models

1. (Programming challenge - generating beautiful simulations of the Heisenberg model (the spin $O(3)$ model))

   Let $n \geqslant 1$ be an integer, let $\beta > 0$ and let $G = (V, E)$ be a finite graph. We simulate the spin $O(n)$ model, at inverse temperature β on the graph G, using a Markov Chain Monte Carlo (MCMC) algorithm. By an MCMC algorithm we mean that we run a Markov chain (σ^k) whose stationary measure is the spin $O(n)$ distribution and output its state σ^K for some sufficiently large K (it takes some experimentation, or theory, to find a value for K for which it appears that the chain has "mixed"). We now describe the Markov chain that will be used, which is called the *Wolff cluster algorithm*:

   (a) Let $\sigma^0 : V \to \mathbb{S}^{n-1}$ be an arbitrary initial configuration.

   (b) For $0 \leqslant k \leqslant K-1$ do:

      i. Sample uniformly at random a vertex $x \in V$.

      ii. Sample uniformly at random a unit vector $s \in \mathbb{S}^{n-1}$.

      iii. Create a random edge configuration $\omega \subseteq E$ by placing each edge $e = \{u, v\} \in E$ into ω with probability $p_s(\sigma_u, \sigma_v)$, independently between edges. The probability $p_s(a, b)$ is given by the formula

      $$p_s(a,b) := \max\left\{0,\ 1 - \exp\left(-2\beta\langle s,a\rangle\langle s,b\rangle\right)\right\} \tag{25}$$

      where $\langle s,t\rangle := \sum_{i=1}^n s_i t_i$ is the standard inner product in $\mathbb{R}^n$.

iv. Let C_x be the connected component of x in the edge configuration ω. Define σ^{k+1} by

$$\sigma^{k+1}(v) := \begin{cases} \sigma^k(v) - 2\langle s, \sigma^k(v)\rangle s & v \in C_x \\ \sigma^k(v) & v \notin C_x \end{cases} \tag{26}$$

(i.e., the spins in C_x are reflected in the hyperplane orthogonal to s).

(c) Output σ^K.

Use the Wolff cluster algorithm to generate samples of the Heisenberg model (the spin $O(3)$ model) on the two-dimensional discrete torus with side length 100 (the graph with vertex set $\{1, 2, \dots, 100\}^2$, where two vertices are adjacent if they differ in exactly one coordinate, and by exactly 1 modulo 100 in that coordinate) - try a few values for β and for each one try to find a good value for K.

The challenge in this problem is to find a good way to visualize the simulation results. That is, to find a visually pleasant mapping from the sphere $\mathbb{S}^2$ to a color palette. A simulation is presented in Figure 3 in the "Lectures on the spin and loop $O(N)$ models" notes. However, the mapping chosen there is visually unsatisfactory since most colors appear to be either red, green or blue, without significant interpolation between these. Find a way to improve this (this may involve experimenting with color palettes and/or searching for ideas on the internet).

Remarks: Good results may be presented on the class webpage with the student's agreement.

In the Ising model case ($n = 1$), the choice of $s \in \{-1, 1\}$ does not affect the formulas (25) and (26) and hence s may be ignored.

For increased efficiency of the algorithm, it suffices to sample at every iteration only those edges of ω which determine the connected component C_x.

More information on the Wolff cluster algorithm and related operations can be found, for instance, in section 2 of the paper "Rarity of extremal edges in random surfaces and other theoretical applications of cluster algorithms".

2. (Localization of 1-Lipschitz integer-valued height functions at low temperature)

Let $d \geqslant 2$ and $L \geqslant 1$ be integers. Let $x > 0$. Let $G = (V, E)$ be the d-dimensional discrete cube of side length $2L + 1$ (the graph with vertex set $\{-L, -L + 1, \dots, L\}^d$ with two vertices adjacent if they differ in exactly one coordinate, and by exactly one in that coordinate). The configuration space of 1-Lipschitz integer-valued height functions with zero boundary conditions is

$$\Omega := \left\{ \phi : V \to \mathbb{Z} : \begin{array}{c} \phi(u) - \phi(v) \in \{-1, 0, 1\} \text{ for } \{u, v\} \in E, \\ \phi(v) = 0 \text{ for } v \text{ with } \|v\|_\infty = L \end{array} \right\}. \tag{27}$$

We place a probability measure μ_x on Ω by setting the probability of each $\phi \in \Omega$ to be proportional to $x^{N(\phi)}$ where

$$N(\phi) := \{\{u, v\} \in E \colon \phi(u) \neq \phi(v)\} \tag{28}$$

is the number of nearest-neighbor pairs where the values of ϕ differ.

Prove that there exists some $x_0 > 0$, which may depend on d but not on L, such that whenever $0 < x < x_0$ then

$$\mu_x(\phi(v) \neq 0) < 0.01 \text{ for all } v \in V. \tag{29}$$

Hint: Find a way to adapt the Peierls argument to this setting.

3. (Delocalization of two-dimensional real-valued height functions)

   Let $U : \mathbb{R} \to \mathbb{R}$ be a twice-continuously differentiable function satisfying

   (a) (even function) $U(x) = U(-x)$ for all $x \in \mathbb{R}$.

   (b) (growth at infinity) $\frac{U(x)}{\log x} \to \infty$ as $x \to \infty$.

   (c) (bounded second derivative) $\sup_{x\in\mathbb{R}} U''(x) < \infty$.

   (one example to have in mind is the function $U(x) = x^2$).

   Let $d = 2$ and let $L \geqslant 1$ be an integer. Let $G = (V, E)$ be the two-dimensional discrete square of side length $2L + 1$ as defined in the previous problem (but now with $d = 2$). The configuration space of real-valued height functions with zero boundary conditions is

$$\Omega := \{\phi : V \to \mathbb{R} \colon \phi(v) = 0 \text{ for } v \text{ with } \|v\|_\infty = L\}. \tag{30}$$

   We place a probability measure μ on Ω by setting the density of each $\phi \in \Omega$ to be proportional to

$$\exp\left(-\sum_{\{u,v\}\in E} U(\phi(u) - \phi(v))\right). \tag{31}$$

   Here, the density is with respect to the natural product Lebesgue measure ($\prod_v d\phi(v)$ over all $v \in V$ with $\|v\|_\infty \neq L$). It is not obvious that (31) can indeed be normalized to be a probability measure but this is ensured by the growth at infinity assumption on U.

   Let ϕ be sampled from μ. Prove that there exists some $c > 0$, which may depend on U but not on L, such that

$$\mathrm{Var}(\phi(0,0)) \geqslant c \log L. \tag{32}$$

   Hint: Adapt the proof of the Mermin–Wagner theorem (no continuous-symmetry breaking in two dimensions) to this setting.